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COFINALITIES OF ELEMENTARY SUBSTRUCTURES OF STRUCTURES ON $\aleph_\omega$

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ABSTRACT

We give some sufficient conditions for $\{a \subseteq \aleph_\omega\colon \mathrm{cf}(\sup(a \cap \aleph_n)) = f(n)$ for $n < \omega\}$ to be stationary. Thus we get consistency e.g. of cases where f has $\geq$ three values attained infinitely often. We use pcf, partition theorems on trees and games.

1. Introduction

Let $0 < n^* < \omega$ and $f\colon X \to n^* + 1$ be a function where $X \subseteq \omega \backslash (n^* + 1)$ is infinite. Let's consider the following set:

$$S_{f.} = \{x \subset \aleph_\omega \colon |x| \leq \aleph_{n^*} \wedge (\forall n \in X)\, \mathrm{cf}(x \cap \alpha_n) = \aleph_{f(n)}\}.$$

The question is whether S_f is stationary in $[\aleph_\omega]^{<\aleph_{n^*+1}}$. The question was first posed by Baumgartner in [Ba]. By a standard result, the above question can also

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be rephrased as a certain transfer property. Namely, S_f is stationary iff for any structure $A = \langle \aleph_\omega, \ldots \rangle$ there's a $B \prec A$ such that $|B| = \aleph_{n^*}$ and for all $n \in X$ we have $\mathrm{cf}(B \cap \aleph_n) = \aleph_{f(n)}$.

Note that if f is eventually constant then S_f is always stationary (see [Ba]). Also, any reasonable "finite variation" of f will preserve the property, i.e., if $n_1^* > n^*$ and $f_1: X \to n_1^* + 1$ agrees with f on $\mathrm{dom}(f)\backslash(n_1^* + 1)$, then S_{f_1} is stationary provided that S_f is. So we are interested in the case that f is not eventually constant. You may wonder how strong the statement that S_f is stationary is. Magidor (unpublished, but close to [Mg3]) has shown that if $f: \omega\backslash 2 \to 2$ assumes the values 0 and 1 alternatively and S_f is stationary (here $n^* = 1$), then there's an inner model with infinitely many measurable cardinals.

Liu has proven (in [Liu]) that under the existence of huge cardinals, it is consistent that S_f is stationary together with GCH for many f's assuming (any) two different values. In this paper, we are going to prove a few results concerning the above question, which are due to the second author except for Proposition 3.1 and a small remark of the first author improving Theorem 4.2.

Theorems here (3.1, 4.2) are the first results for function f with more than 2 values gotten infinitely many times. Also the present results have relatively small consistent strength. A version of 4.1 was proved in the mid-eighties and forgotten and we thank M. Gitik for his reminder. The main results are as follows:

THEOREM 3.1: *Assume* $\sup(\mathrm{pcf}(\{\aleph_n: n < \omega\})) = \aleph_{\omega+n^*}$. *Let* $1 < m^* < \omega$. *Let I be the ideal of finite subsets of ω. Let $\langle A_i: 1 \le i \le n^* \rangle$ be pairwise disjoint subsets of $\omega\backslash(m^*+1)$ such that $\prod_{k\in A_i} \aleph_k/I$ has true cofinality $\aleph_{\omega+i}$ for $1 \le i \le n^*$. Let $\langle \kappa_i: 1 \le i \le n^* \rangle$ be a sequence of uncountable cardinals below $\aleph_{m^*+1}$. Then the set*

$$S = \{x \subseteq \aleph_\omega: |x| \le \aleph_{m^*} \wedge (\forall i)[1 \le i \le n^* \to (\forall k \in A_i)\, \mathrm{cf}(x \cap \aleph_\kappa) = \kappa_i]\}$$

is stationary in $[\aleph_\omega]^{<\aleph_{m^*+1}}$.

THEOREM 4.1: *Assume $A \subseteq \omega$, $0 < n^* < \min(A)$ and for each $n \in A$ there is an $\aleph_n$-complete filter F_n on $\aleph_n$ such that F_n contains the cobounded subsets of $\aleph_n$ and the second player has a winning strategy in the game $GM_{\aleph_{n^*}}(F_n)$. (See Definition 4.1.)*

Then the set

$$S = \{x \subset \aleph_\omega: |x| \le \aleph_{n^*} \text{ and } (\forall n \in A)[\mathrm{cf}(x \cap \aleph_n) = \aleph_0] \text{ and }$$
$$(\forall n)[n < \omega \wedge n^* < n \notin A \to \mathrm{cf}(x \cap \aleph_n) = \aleph_{n^*}]\}$$

is stationary in $[\aleph_\omega]^{<\aleph_{n^*+1}}$.

THEOREM 4.2: *Assume* GCH. *Let* $0 \le m < n^* < \omega$ *and* $E \subseteq \omega \backslash n^*$ *be such that for all* $i \in E$, $i+1 \notin E$. *Let* $\langle n_i : i \in E\rangle$ *with each* $n_i < n^*$. *Suppose that for each* $i \in E$, *there is an* $\aleph_i$*-complete filter* F_i *on* $\aleph_i$ *containing all clubs of* $\aleph_i$ *such that* $W_i = \{\alpha < \aleph_i : \mathrm{cf}(\alpha) = \aleph_{n_i}\} \in F_i^+$ *and the second player has a winning strategy in the game* $GM'_{\aleph_m}(F_i)$ *(see Definition 4.1).*

Let $f: \omega \to n^*$ *be the function defined by*

$$f(i) = \begin{cases} n_i & \text{if} \in E, \\ m & \text{if } n^* < i \notin E. \end{cases}$$

Then the set $S' = \{x \subseteq \aleph_\omega : |x| \le \aleph_{n^*} \text{ and } (\forall i > n^*)[\mathrm{cf}(x \cap \aleph_i) = \aleph_{f(i)}]\}$ *is stationary in* $[\aleph_\omega]^{<\aleph_{n^*+1}}$.

In this paper, we concentrate on $\{\aleph_n : n \in \omega\}$, but we can deal with other sets with natural changes (at the referee's request an explanation was added in 95 at the end of the paper). We implicitly assume that all models under consideration have a countable language.

2. Preliminaries

Let's start with a standard result. The proof will be omitted.

PROPOSITION 2.1: *Let* $n^* < \omega$, $X \subseteq \omega \backslash (n^*+1)$ *and* $f: X \to n^*+1$. *Consider the set* $S = S_f$ *as defined in Section 1. Let* $\theta > (\aleph_\omega^{\aleph_{n^*}})^+$ *be a regular cardinal. Let* $M \prec \langle \mathcal{H}_\theta, \in, S, \triangleleft, \ldots\rangle$, $M \supseteq \aleph_\omega + 1$ *and* $|M| = \aleph_\omega$, *where* $\triangleleft$ *is a well-ordering of* $\mathcal{H}_\theta$. *Then the following are equivalent:*

1. S *is stationary in* $[\aleph_\omega]^{<\aleph_{n^*+1}}$.
2. *For any structure* $A = \langle \aleph_\omega, \ldots\rangle$ *with a countable language, there is a* $B \prec A$ *such that* $|B| = \aleph_{n^*}$ *and* $B \in S$.
3. *There is* $N \prec M$ *such that* $|N| = \aleph_{n^*}$ *and* $\forall n \in X$, $\mathrm{cf}(N \cap \aleph_n) = \aleph_{f(n)}$.

LEMMA 2.1: *Let* $\kappa < \mu < \lambda$ *be regular cardinals. Let* $A = \langle \mathcal{H}_\lambda, \in, \triangleleft, \kappa, \mu, \ldots\rangle$ *be a structure of a countable language on* $\mathcal{H}_\lambda$ *with skolem functions closed under composition and* $\triangleleft$ *a well-ordering of* $\mathcal{H}_\lambda$. *If* $B \prec A$ *and* $X \subseteq \kappa$ *and* $B' = sk^A(B \cup X)$, *then* $\sup(B' \cap \mu) = \sup(B \cap \mu)$.

Proof: It's clear that the lemma holds if $\sup(B \cap \mu) = \mu$. So we assume $\sup(B \cap \mu) < \mu$.

It's clear that $\sup(B \cap \mu) \leq \sup(B' \cap \mu)$. Now suppose $\alpha \in B' \cap \mu$. WLOG, assume $\alpha = \tau(\vec{b}, x_0)$ for some $\vec{b} \in [B]^{<\omega}$, $x_0 \in X$ and some skolem function τ. Define $f\colon [\kappa]^{<\omega} \to \mu$ by letting $f(x) = \tau(\vec{b}, x)$ if $\tau(b, x) < \mu$ and $f(x) = 0$ otherwise. Then f is definable from $\vec{b}$ in B. So $f \in B$. Let $\delta = \sup(f''[\kappa]^{<\omega})$. Then $\delta \in B \cap \mu$ and $B' \models \delta = \sup(f''[\kappa]^{<\omega})$. So $B' \models \alpha = f(x_0) \leq \delta$. Hence $\alpha \leq \delta < \sup(B \cap \mu)$. Therefore, $\sup(B \cap \mu) \geq \sup(B' \cap \mu)$. This completes the proof of the lemma. ∎

LEMMA 2.2: *Let $n \in \omega$ and $X_0, X_1, \ldots, X_{n-1}$ be disjoint subsets of ω. Let $X = \bigcup_{i<n} X_i$ and f be a function from X to ω such that f is constant on X_i for each $i < n$ and the constant values of f on different X_i's are distinct. Let $A = \langle \aleph_\omega, \ldots \rangle$ be an algebra on $\aleph_\omega$. Let $B \prec A$ be such that $(\forall i < n)\ (\forall m \in X_i)$ $[\mathrm{cf}(B \cap \aleph_m) = \aleph_{f(m)}]$. Let $k = \max(f''X)$, $A_{i_0} = f^{-1}\{k\}$ and $\ell < k$. Then for any n^* such that $n^* \geq \ell$ and $n^* > \max(f''X \backslash \{k\})$, and $j < \omega$ such that $|B \cap \aleph_j| = \aleph_{n^*}$, there is $B' \prec B$ such that*

(1) $|B'| = \aleph_{n^*}$ *and* $B' \cap \aleph_m = B \cap \aleph_m$ *for* $m \leq j$;

(2) $(\forall m \in X_{i_0})\ [m > j \to \mathrm{cf}(B' \cap \aleph_m) = \aleph_\ell]$;

(3) $(\forall m \in X \backslash X_{i_0})\ [m > j \to \mathrm{cf}(B' \cap \aleph_m) = \mathrm{cf}(B \cap \aleph_m)]$.

Proof: For each $i \neq i_0$, for each $m \in X_i$, let a_m be a cofinal subset of $B \cap \aleph_m$ with order type $\aleph_{f(i)}$. Now we can build a sequence $\langle B_\alpha\colon \alpha < \aleph_\ell \rangle$ such that

(1) $\bigcup\{a_m\colon m \in X \backslash X_{i_0}\} \cup (B \cap \aleph_j) \subseteq B_0$;

(2) $\forall m \in X_{i_0} \setminus j[\sup(B_\alpha \cap \aleph_m) < \sup(B_{\alpha+1} \cap \aleph_m)]$;

(3) $B_\alpha \prec B_{\alpha+1} \prec B$ and $|B_\alpha| = \aleph_{n^*}$.

The construction is obvious. Now let $B' = \bigcup_{\alpha < \aleph_\ell} B_\alpha$. It is clear that B' is as required. ∎

3. An application of pcf theory

We are going to prove the following theorem using pcf theory (see [Sh:g]).

THEOREM 3.1: *Assume $\max(\mathrm{pcf}(\{\aleph_n\colon n < \omega\})) = \aleph_{\omega+n^*}$. Let $1 < m^* < \omega$. Let I be the ideal of finite subsets of ω. Let $\langle A_i\colon 1 \leq i \leq n^* \rangle$ be a sequence of pairwise disjoint subsets of $\omega \backslash (m^*+1)$ such that $\prod_{k \in A_i} \aleph_k / I$ has true cofinality $\aleph_{\omega+i}$ for $1 \leq i \leq n^*$. Let $\langle \kappa_i\colon 1 \leq i \leq n^* \rangle$ be a sequence of uncountable cardinals below $\aleph_{m^*+1}$. Then the set*

$$S = \{x \subseteq \aleph_\omega\colon |x| \leq \aleph_{m^*} \wedge (\forall i)(1 \leq i \leq n^* \to (\forall k \in A_i)[\mathrm{cf}(x \cap \aleph_k) = \kappa_i])\}$$

is stationary in $[\aleph_\omega]^{<\aleph_{m^*+1}}$.

Remarks:

1. Using Lemma 2.2, Lemma 2.1 and this theorem, we can show that for any given $1 \le n \in \omega$ and $\langle \kappa_i : 1 \le i \le n^* \rangle$ with cardinals $\kappa_i \le \aleph_n$ for $1 \le i \le n^*$, we have that the set

$$S' = \{x \subset \aleph_\omega : |x| \le \aleph_n \text{ and } (\forall i)(1 \le i \le n^* \rightarrow \forall k \in A_i[\mathrm{cf}(x \cap \aleph_i) = \kappa_i])\}$$

is stationary in $[\aleph_\omega]^{<\aleph_{n+1}}$. Also we can weaken the demand "$A_i \subseteq \omega \setminus (m^* + 1)$" to "$A_i \subseteq \omega \setminus (\omega + 1)$".
2. Notice that in order for the theorem not to be trivial, we assume $n^* > 1$ and therefore GCH fails at $\aleph_\omega$.
3. If $\mathrm{pp}\,\aleph_\omega \underset{\mathrm{def}}{=} \sup(\mathrm{pcf}(\{\aleph_m : m < \omega\})) > \aleph_{\omega+n^*}$, no harm is done since we can use Levy collapse to collapse $\mathrm{pp}\,\aleph_\omega$ to $\aleph_{\omega+n^*}$ and no new subset of $\aleph_\omega$ is added.
4. The theorem can be generalized to other singular cardinals. Also, we can use other regular cardinals in $(\aleph_\omega, \mathrm{pp}\,\aleph_\omega)$ in the proof of the theorem.
5. Consistency results giving the assumptions are well known, starting with [Mg]; see history and references on this in [Sh:g].

The proof of Theorem 3.1 uses the following lemma:

LEMMA 3.1: *For each* $1 \le i \le n^*$, *there is a sequence* $\vec{C}^i \underset{\mathrm{def}}{=} \langle C^i_\alpha : \alpha < \aleph_{\omega+i} \rangle$ *such that*

(1) $\forall \alpha C^i_\alpha \subseteq \alpha$ *and* $o.t.(C^i_\alpha) \le \kappa_i$,
(2) $\beta \in C^i_\alpha$ *implies* $C^i_\beta = C^i_\alpha \cap \beta$,
(3) $S_i \underset{\mathrm{def}}{=} \{\alpha < \aleph_{\omega+i} : \mathrm{cf}(\alpha) = \kappa_i$ *and* $\alpha = \sup(C^i_\alpha)\}$ *is stationary in* $\aleph_{\omega+i}$.

Remarks: Note that the C^i_α's are not necessarily closed.

For a proof of Lemma 3.1, see [Sh 351, 4.4] for successor of regular cardinals and in general [Sh 420, 1.5] which rely on [Sh 351, 4.4].

For each $1 \le i \le n^*$, let S_i and $\vec{C}^i = \langle C^i_\alpha : \alpha < \aleph_{\omega+i} \rangle$ be as in Lemma 3.1. We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1: For $1 \le i \le n^*$, let $\vec{f^i} = \langle f^i_\alpha : \alpha < \aleph_{\omega+i} \rangle \subseteq \Pi_{k \in A_i} \aleph_k$ be increasing and cofinal in $\Pi_{k \in A_i} \aleph_k / I$. Let $\lambda >> \mathrm{pp}\,\aleph_\omega$ be a regular cardinal. Let's consider the structure $A = \langle \mathcal{H}_\lambda, \in, \triangleleft, \ldots \rangle$ with skolem functions closed under composition, where $\triangleleft$ is a well-ordering on $\mathcal{H}_\lambda$. We define $X_{\vec{\alpha}}, N_{\vec{\alpha}}$

by induction on $\vec{\alpha}_{n^*}$ as follows. Let $x = \{\vec{f^i}, A_i, \vec{C^i}\}_{1\le i\le n^*}$. For each $\vec{\alpha} = \langle \alpha_i: 1 \le i \le n^*\rangle \in \Pi_{1\le i\le n^*}\aleph_{\omega+i}$, let

$$X_{\vec{\alpha}} = \{\gamma: \gamma < \aleph_{n^*}\} \cup (\bigcup_{1\le i\le n^*} C^i_{\alpha_i}) \cup \{N_{\vec{\beta}}: \vec{\beta} \in \prod_{i=1}^{n} C^i_{\alpha_i}\} \cup x$$

$$(\text{hence } C^i_\xi \subseteq X_{\vec{\alpha}} \text{ if} : 1 \le i \le n^* \wedge \xi \in C^i_{\alpha_i})$$

and let $N_{\vec{\alpha}} = sk^A(X_{\vec{\alpha}})$. Note that $|N_{\vec{\alpha}}| = \aleph_n$ and $\vec{\alpha} \in \prod_{i=1}^{n^*} C^i_{\beta_i} \Rightarrow N_{\vec{\alpha}} \prec N_{\vec{\beta}}$ and $N_{\vec{\alpha}} \in N_{\vec{\beta}}$.

Claim: There is $\vec{\delta} = \langle \delta_i: 1 \le i \le n^*\rangle \in \Pi_{1\le i\le n^*} S_i$ such that for all $1 \le i \le n^*$:

(1) For all $\vec{\alpha} \in \Pi_{1\le i\le n^*} C^i_{\delta_i}$, we have $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) < \delta_i$ for all $1 \le i < n^*$.

(2) $\sup(N_{\vec{\delta}} \cap \aleph_{\omega+i}) = \delta_i$.

(3) For some $n_i < \omega$, $\{f^i_\alpha(k): \alpha \in C^i_{\delta_i}\}$ is cofinal in $N_{\vec{\delta}} \cap \aleph_k$ for all $k \in A_i \backslash n_i$.

(4) For some $m_i \ge n_i$, $\mathrm{cf}(N_{\vec{\delta}} \cap \aleph_k) = \mathrm{cf}(\delta_i) = \kappa_i$ for all $k \in A_i \backslash m_i$.

Proof of Claim: We first construct $\vec{\delta}$ as required by part (1) of the claim. The construction is as follows: Let $E_{n^*} = \{\delta < \aleph_{\omega+n^*}: \forall\vec{\alpha} \in \Pi_{1\le i\le n^*}\aleph_{\omega+i}$, if $\alpha_{n^*} < \delta$, then $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+n^*}) < \delta\}$. Then E_{n^*} is clearly closed unbounded in $\aleph_{\omega+n^*}$. Since S_{n^*} is stationary in $\aleph_{\omega+n^*}$ we have $S_{n^*} \cap E_{n^*} \ne \emptyset$. Pick some $\delta_{n^*} \in S_{n^*} \cap E_{n^*}$.

Suppose we have defined δ_j for $n^* \ge j > i$. We now define δ_i. Let $E_i = \{\delta < \aleph_{\omega+i}: \forall\vec{\alpha} \in \Pi_{1\le \ell\le n^*}\aleph_{\omega+\ell}$ if $\alpha_i < \delta$ and $\alpha_j = \delta_j$ for all $i < j \le n^*$ then $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) < \delta\}$. It's easy to see that E_i is closed unbounded in $\aleph_{\omega+i}$. So we can find $\delta_i \in S_i \cap E_i$ since S_i is stationary in $\aleph_{\omega+i}$.

We now show $\vec{\delta}$ satisfies clause (1). Let $\vec{\alpha} \in \Pi_{1\le i\le n^*} C^i_{\delta_i}$. Let $1 \le i \le n^*$ be fixed, and we want to show that $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) < \delta_i$. Consider $\vec{\beta} = \langle \alpha_1, \ldots \alpha_i, \delta_{i+1}, \ldots \delta_{n^*}\rangle$. By the choice of δ_i, we have that $\sup(N_{\vec{\beta}} \cap \aleph_{\omega+i}) < \delta_i$. Since $(\forall j)[1 \le j \le n^* \to C^j_{\alpha_j} \subseteq C^j_{\beta_j}]$ clearly $X_{\vec{\alpha}} \subseteq X_{\vec{\beta}}$. So $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) \le \sup(N_{\vec{\beta}} \cap \aleph_{\omega+i}) < \delta_i$.

Let's now prove clause (2) of the claim. Fix $1 \le i \le n^*$. Let $\beta \in N_{\vec{\delta}} \cap \aleph_{\omega+i}$. Then $\beta = \tau(\vec{y})$ for some $\vec{y} \in [X_{\vec{\delta}}]^{<\omega}$ and some skolem function τ. We need to show $\beta < \delta_i$. Since $\delta_j = \sup(C^j_{\delta_j})$ and $C^j_{\delta_j}$ has no last element for $1 \le j \le n^*$, there is $\vec{\alpha} \in \Pi_{1\le j\le n^*} C^j_{\delta_j}$ such that $\vec{y} \in [X_{\vec{\alpha}}]^{<\omega}$. But then $\beta \in N_{\vec{\alpha}} \cap \aleph_{\omega+i}$. By clause (1) we have $\beta < \delta_i$, so $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+1}) \le \delta_i$ and, as $C^i_{\delta_i} \subseteq N_{\vec{\delta}}$ and $C^i_{\delta_i} \subseteq \delta_i = \sup(C^i_{\delta_1})$ because $\delta_i \in S_i$, we get equality.

We show clause (3) by contradiction. Assume that (3) fails. So there is an unbounded set $b \subseteq A_i$ such that $\forall k \in b \exists \beta_k \in N_{\vec{\delta}} \cap \aleph_k [\sup(\{f^i_\xi(k): \xi \in C^i_{\delta_i}\}) < \beta_k]$. Fix such β_k for each $k \in b$. Since for each $1 \leq j \leq n^*$ the set $C^j_{\delta_j}$ has order whose cofinality is uncountable, there is $\vec{\alpha} \in \Pi_{1 \leq \ell \leq n^*} C^\ell_{\delta_\ell}$ such that $(\forall k \in b)\beta_k \in N_{\vec{\alpha}}$. Clearly $k \in b \Rightarrow \beta_k < \sup(N_{\vec{\alpha}} \cap \aleph_k)$ hence $\langle \beta_k : k \in b \rangle < \langle \sup(N_{\vec{\alpha}} \cap \aleph_k): k \in b\rangle$. The latter belong to $\Pi_{k \in b} \aleph_k$ hence for some $\xi < \aleph_{\omega+i}$ we have $\langle \sup(N_{\vec{\alpha}} \cap \aleph_k): k \in b\rangle \leq^* f^i_\xi$, where $f \leq^* g$ means $f(k) \leq g(k)$ for all but finitely many k's. Since $N_{\vec{\alpha}} \in N_{\vec{\delta}}$, clearly $\langle \sup(N_{\vec{\alpha}} \cap \aleph_k): k \in b\rangle$ belongs to $N_{\vec{\delta}}$ and also $\langle f^i_\xi : \xi < \aleph_{\omega+i}\rangle$ belongs to $X_{\vec{\delta}}$ hence to $N_{\vec{\delta}}$, so wlog $\xi \in N_{\vec{\delta}} \cap \aleph_{\omega+i}$. Now we can replace ξ by any $\xi' \in (\xi, \aleph_{\omega+1})$ and $C^i_{\delta_i}$ is unbounded in $N_{\vec{\delta}} \cap \aleph_{\omega+i}$, so wlog $\xi \in C^i_{\delta_i}$ hence $k \in b \Rightarrow f_\xi(k) \in N_{\vec{\delta}} \cap \aleph_{\omega+i}$ hence is $< \beta_k$. This is clearly absurd.

Finally, let's prove clause (4) again by contradiction. Suppose clause (4) is not true. Then there is an unbounded set $b \subseteq A_i \backslash n_i$ such that $\mathrm{cf}(\sup\{f^i_\xi(k): \xi \in C^i_{\delta_i}\}) < \kappa_i$ for some $k \in b$. So as $\mathrm{o.t.}(C^i_{\delta_i}) = \kappa_i$ for each $k \in b$, there is $\xi_k \in C^i_{\delta_i}$ such that for all $\xi \in C^i_{\delta_i}$ with $\xi \geq \xi_k$ we have: $\sup\{f^i_\zeta(k): \zeta \in C^i_{\delta_i}\} = \sup\{f^i_\zeta(k): \zeta \in C^i_{\delta_i} \cap \xi\}$. Let $\beta \in C^i_{\delta_i}$ be such that $\beta > \sup(\{\xi_k : k \in b\})$. For each $k \in A_i$, let $\mu_k = \sup(\{f^i_\zeta(k): \zeta \in C^i_\beta\})$. Then $\langle \mu_k : k \in A_i\rangle \in N_{\vec{\delta}} \cap \Pi_{k \in A_i} \aleph_k$ since $C^i_\beta \in N_{\vec{\delta}}$. So as above there is $\beta' \in N_{\vec{\delta}} \cap \aleph_{\omega+i}$ such that $\langle \mu_k : k \in A_i\rangle \leq^* f^i_{\beta'}$. So we have $\langle \sup(N_{\vec{\delta}} \cap \aleph_k): k \in b\rangle = \langle \mu_k : k \in b\rangle \leq^* f^i_{\beta'} \restriction b$ which contradicts $f^i_{\beta'} \in N_{\vec{\delta}} \cap \Pi_{k \in A_i} \aleph_k$ (the initiated could have used "wlog $\vec{f}^i$ obeys $\vec{C}^i$"). This completes the proof of the claim.

Now, Theorem 3.1 follows from the claim, Proposition 2.1 and Lemma 2.1. ∎

By the remarks following Theorem 3.1, the theorem is not trivial only when $\mathrm{pp}\,\aleph_\omega > \aleph_{\omega+1}$. In particular GCH does not hold at $\aleph_\omega$. But using the following observation, we can make GCH hold at $\aleph_\omega$ by collapsing $2^{\aleph_\omega}$ and still have the desired conclusion in the forcing extension, i.e., the set S in Theorem 3.1 is still stationary in the generic extension. (So by well-known consistency results we can even have GCH.)

PROPOSITION 3.1: *Let P be an $< \aleph_{\omega+1}$-closed forcing notion. Suppose S is stationary in $[\aleph_\omega]^{<\aleph_{n+1}}$ in V. Then $V^P \Vdash$ "S is stationary in $[\aleph_\omega]^{<\aleph_{n+1}}$".*

Proof: It suffices to show that in V^P, for any given structure $A = \langle \aleph_\omega, \dots\rangle$ of a countable language, there is $B \prec A$ such that $|B| = \aleph_n$ and $B \in S$.

Let $p \in P$ for that $\dot{A} = \langle \aleph_\omega, (\dot{f}_i)_{i \in \omega}\rangle$ is a structure on $\aleph_\omega$ with skolem functions

$\dot{f}_i$ closed under compositions.

Since P is $(< \aleph_{\omega+1})$-closed, we can find $\langle p_\alpha : \alpha < \aleph_\omega \rangle$ such that p_α is stronger than p and p_β for $\beta < \alpha$ and f_i' in V such that for each i, for any $\vec{\alpha} \in [\aleph_\omega]^{<\omega}$ there is a β such that $p_\beta \Vdash \dot{f}_i(\vec{\alpha}) = f_i'(\vec{\alpha})$ whenever $\vec{\alpha} \in \mathrm{dom}(\dot{f})$.

Consider $A' = \langle \aleph_\omega, (f_i')_{i \in \omega} \rangle$ in V. Let p be such that p is stronger than p_α for all $\alpha < \aleph_\omega$. Let $B \prec A'$ be such that $|B| = \aleph_n$ and $B \in S$. But then $p \Vdash B \prec \dot{A}$ since $p \Vdash f_i' = \dot{f}_i$. This is as required. ∎

4. Applications of large ideals

In this section, we prove two results under the existence of large ideals (on the $\aleph_n$'s). Before we state our results, we need some terminology.

Definition 4.1: Let $\kappa > \lambda$ be cardinals. Let D be a filter on κ.
(1) We define the game $\underline{GM_\lambda(D)}$ as follows: the game lasts λ moves. At the ξ^{th} move, the first player chooses a subset A_ξ of κ such that $A_\xi \subseteq \bigcap_{\eta<\xi} B_\eta$, and if $\bigcap_{\eta<\xi} B_\eta \neq \emptyset \,\mathrm{mod}(D)$ then $A_\xi \neq \emptyset \,\mathrm{mod}(D)$. The second player chooses a subset B_ξ of A_ξ with $B_\xi \neq \mathrm{mod}(D)$.

A player without a legal move loses the game immediately. (Note this can only happen to the second player.) If the game lasts for λ moves, the second player wins if $\bigcap_{\xi<\lambda} B_\xi$ is unbounded.
(2) Let's also assume that D is κ-complete. We now define the "cut-and-choose" game $\underline{GM'_{\lambda,\kappa}(D)}$ of length λ: at the 0^{th} move, the first player chooses a set $A_0 \neq \emptyset \,\mathrm{mod}(D)$ and then partitions A_0 into less than κ parts; the second player chooses one of the parts, say $B_0 \in D^+$. At the ξ^{th} move for $\xi > 0$, the first player partitions the set $\bigcap_{\eta<\xi} B_\eta$ into less than κ parts, and the second player chooses one of the parts, call it B_ξ such that $B_\xi \in D^+$.

The winning condition for each player is exactly as in the game defined in part (1) above.
(3) Let D be a filter on κ, Θ a set of regular cardinals $< \lambda$. The game $GM^1_\lambda(D, \Theta)$ has λ moves. At the 0th move the first player chooses a set $A_0 \neq \emptyset \,\mathrm{mod}(D)$ and then choose $\kappa_0 \in \Theta$ and $f_0 \colon A_0 \to \kappa_0$, the second player chooses $B_0 = \{\alpha \in A_0 \colon f_0(\alpha) < \zeta_0\}$ for some $\zeta_0 < \kappa_0$, such that $B_0 \in D^+$. At the ξth-move for $\xi > 0$ the first player chooses $\kappa_\xi \in \Theta$ and $f_\xi \colon \cap_{\eta<\xi} B_\eta \to \kappa_\xi$ and the second player chooses $B_\xi = \{\alpha \in \cap_{\eta<\xi} B_\eta \colon f_\xi(\alpha) < \zeta_\xi\}$ for some $\zeta_\xi < \kappa_\xi$ such that $B_\xi \in D^+$. The winning condition is as above.

Let's first prove the following theorem:

THEOREM 4.1: *Assume $A \subseteq \omega, 0 < n^* < \min(A)$ and for each $n \in A$ there is a filter F_n on $\aleph_n$ such that F_n contains the cobounded subsets of $\aleph_n$ and the second player has a winning strategy in the game $GM_{\aleph_{n^*}}(F_n)$.*

Then the set

$$S = \{x \subset \aleph_\omega : |x| \leq \aleph_{n^*}, (\forall n \in A)\, \mathrm{cf}(x \cap \aleph_n) = \aleph_0 \quad \text{and} \quad (\forall n)[\forall n^* < n \notin A \rightarrow \mathrm{cf}(x \cap \aleph_n) = \aleph_{n^*}]\}$$

is stationary in $[\aleph_\omega]^{<\aleph_{n^*+1}}$.

Remarks: (1) Before we prove Theorem 4.1, we would like to see how we can get the filters as required in the hypothesis of the theorem. Magidor (see [Mg1]) has shown the consistency of the existence of the filters. Also, Laver has proved that if we collapse a measurable cardinal κ to some $\aleph_n$, then in the generic extension, there is a normal ideal on F_n on $\aleph_n$ such that $\mathcal{P}(\aleph_n)/F_n$ has a $< \aleph_{n-1}$-closed dense subset. Therefore, if there are infinitely many measurable cardinals, say $\langle \kappa_n : n < \omega \rangle$, $A = \{m_m | n < \omega\}$, $n^* < m_n < n$, $m_n + 1 < m_{n+1}$ we can Levy collapse each κ_n to $\aleph_{n^*+m_n}$ to get the normal filters as required in the hypothesis of Theorem 4.1.

(2) We can also use (in the assumption of Theorem 4.1) the games $GM'_{\aleph_{n^*}, \aleph_n}(F_n)$ in place of $GM_{\aleph_{n^*}}(F_n)$. We can weaken it further using, for $n \in A$, the following game for F_n (see [Sh 250]) (see better [Sh:f, Ch XIV]): in the ξ^{th} move, player one chooses $m_\xi \in \omega \backslash A, n^* < m_\xi < n$ and $F_\xi : \aleph_n \rightarrow \aleph_{m_\xi}$ and player two has to choose $B_\xi \subseteq \cap_{\zeta<\xi} B_\xi \cap A_0$ such that the range of $f_\xi \restriction B_\xi$ is bounded in $\aleph_{m_\xi}$ and $B_\xi \neq \emptyset \bmod F_n$; in the 0^{th} move player one also chooses $A_0 \subseteq \aleph_{m_\xi}, A_0 \neq \emptyset \bmod F_n$.

Proof: In order to prove Theorem 4.1, let's consider tagged trees of the form $\langle T, I \rangle$, which by definition means that

1. $T \subset [ON]^{<\omega}$ is a tree, i.e. T consists of finite sequences of ordinals closed under initial segments.
2. $\mathcal{I} = \langle I_\sigma : \sigma \in T \rangle$ is such that for each $\sigma \in T, I_\sigma$ is an ideal on $\mathrm{Suc}_T(\sigma)$ which is the set of immediate successors of σ in T. Also, I_σ can be thought of as an ideal on $\{\alpha : \sigma^1 \langle \alpha \rangle \in T\}$.

If T_1 is a subtree of T, we can view $\langle T_1, \mathcal{I}' \rangle$ as a tagged tree with $\mathcal{I}' = \langle I_\sigma | \mathrm{Suc}_{T_1}(\sigma) : \sigma \in T_1 \rangle$. By abuse of notation, we still denote it by $\langle T_1, \mathcal{I} \rangle$. If the family $\mathcal{I}$ of ideals is clear from the context, we will simply say T is a tagged tree without mentioning $\mathcal{I}$ explicitly.

For $X \subseteq T$, let $T[X] = \{\sigma \in T\colon \exists \eta \in X(\eta \leq_T \sigma \vee \sigma \leq_T \eta)\}$. Clearly $T[X]$ is a subtree of T. The following lemma is from [RuSh 117] or [Sh:b, Ch. X] or [Sh:f, Ch. X] and we will not give the proof here.

LEMMA 4.1: *Let $\langle T, \mathcal{I}\rangle$ be a tagged tree such that for each $\sigma \in T$, I_σ is a proper ideal such that $\mathrm{Suc}_T(\sigma) \notin I_\sigma$. Let λ be a regular, uncountable cardinal and for every $\sigma \in T$, I_σ is λ-indecomposable, i.e. if $A \subseteq \mathrm{Suc}_T(\sigma) A \neq \emptyset \bmod I_\sigma$ and $f\colon A \to \lambda$ then for some $\zeta < \lambda$ we have $\{x \in A\colon f(x) < \zeta\} \neq \emptyset \bmod I_\sigma$. (This holds if for every $\sigma \in T$, I_σ is a λ^+-complete ideal or $|\mathrm{Suc}_T(\sigma)| < \lambda$.) Then*

(*) *for every function $F\colon T \to \lambda$, there is a subtree T_1 of T such that for all $\sigma \in T_1$, $\mathrm{Suc}_{T_1}(\sigma) \notin I_\sigma$ and $\mathrm{Sup}(F''T_1) < \lambda$.*

We now proceed to prove Theorem 4.1.

Proof of Theorem 4.1: Let $\langle m_i\colon i < \omega\rangle$ be such that each $m_i \in A$ and for each $m \in A$ there are infinitely many i with $m = m_i$. Let $\mathcal{T} = \{\, T\colon T$ is a subtree of $\bigcup_{l<\omega} \prod_{i<l} \aleph_{m_i}$ and $\forall \eta \in T$ with $lh(\eta) = i$ we have $\{\, \alpha < \aleph_{m_i}\colon \eta\hat{\ }\langle\alpha\rangle \in T \,\} \in F^+_{m_i} \,\}$, where $lh(\eta)$ means the length of the finite sequence η.

Note that each $T \in \mathcal{T}$ can be considered as a tagged tree where for each $\sigma \in T$, the associated ideal I_σ is just the dual ideal to the filter $F_{m_{lh(\sigma)}}$.

Suppose $A = \langle \aleph_\omega, \ldots\rangle$ is an arbitrary structure on $\aleph_\omega$. We are going to find a $B \prec A$ such that $|B| = \aleph_{n^*}$ and for each $n^* < m < \omega$ if $m \in A$ then $\mathrm{cf}(B \cap \aleph_m) = \aleph_0$; if $m \notin A$ then $\mathrm{cf}(B \cap \aleph_m) = \aleph_{n^*}$. This is enough to prove the theorem by Proposition 2.1.

By induction on $\xi < \aleph_{n^*}$, we are going to build $T_\xi, \langle \alpha_{\xi,m}\colon n^* < m \notin A\rangle$, $\langle A_{\xi,\eta}, B_{\xi,\eta}\colon \eta \in T_{\xi+1}\rangle$ and $\langle N_{\xi,\eta}\colon \eta \in T_\xi\rangle$ such that

1. $T_\xi \in \mathcal{T}$ and for any $\xi < \xi', T_{\xi'} \subseteq T_\xi$.
2. For $\eta \in T_{\xi+1}$, $B_{\xi,\eta} = \{\alpha < \aleph_{m_{lh(\eta)}}\colon \eta\hat{\ }\langle\alpha\rangle \in T_{\xi+1}\}$. Furthermore, $\langle A_{\xi',\eta}, B_{\xi',\eta}\colon \xi' \leq \xi\rangle$ is an initial segment of a play of the game $GM_{\aleph_{n^*}}(F_{m_{lh(\eta)}})$ with the second player following his winning strategy.
3. If ξ is a limit ordinal, $T_\xi = \bigcap_{\xi'<\xi} T_{\xi'}$.
4. For $\eta \in T_\xi, N_{\xi,\eta} = \mathrm{sk}^A(\mathrm{ran}(\eta) \cap \{\alpha_{\xi',m}\colon \xi' < \xi$ and $n^* < m \notin A\})$.
5. For $\eta \in T_{\xi+1}$ and $n^* < m \notin A$, we have $\mathrm{Sup}(N_{\xi+1,\eta} \cap \aleph_m) < \alpha_{\xi+1,m}$.
6. For each $n^* < m \notin A$, $\langle \alpha_{\xi,m}\colon \xi < \aleph_{n^*}\rangle$ is an increasing sequence of ordinals in $\aleph_m$.

Take any $T_0 \in \mathcal{T}$ to start with. For ξ limit, let $T_\xi = \cap_{\xi' < \xi} T_{\xi'}$. Since the second player has a winning strategy in the game $GM_{\aleph_{n^*}}(F_m)$ for each $m \in A$, $T_\xi \in \mathcal{T}$ for ξ limit.

If $\xi = 0$, we let $\alpha_{\xi,m} = 0$ for $n^* < m \notin A$. If ξ is limit, we let $\alpha_{\xi,m} = \mathrm{Sup}(\{\alpha_{\xi',m}: \xi' < \xi\})$ for $n^* < m \notin A$.

Suppose T_ξ and $\langle \alpha_{\xi,m}: n^* < m \notin A\rangle$ have been constructed. We now construct $T_{\xi+1}, \langle \alpha_{\xi+1,m}: n^* < m \notin A\rangle$ and $\langle A_{\xi,n}, B_{\xi,\eta}: \eta \in T_{\xi+1}\rangle$.

Let $\langle k_i: i < \omega\rangle$ be an enumeration of $\{m \notin A: n^* < m < \omega\}$. We will define $\langle T'_i: i < \omega\rangle, \langle A_{i,\eta}, B_{i,\eta}: i < \omega\rangle$ for $\eta \in T'_{i+1}$ and $\langle \alpha_i: i < \omega\rangle$ by induction on i such that

1. for each i, $T'_{i+1} \subseteq T'_i \in \mathcal{T}$ and $\mathrm{Sup}(N_{\xi+1,\eta} \cap \aleph_{k_i}) < \alpha_i$ for $\eta \in T'_{i+1}$;
2. $\langle A_{i,\eta}, B_{i,\eta}: i \in \omega\rangle$ is an initial segment of the play of the game $GM_{\aleph_{n^*}}(F_{lh(\eta)})$ with player two following his winning strategy.

Let $T'_0 = T_\xi$. Suppose we have defined T'_i, α_{i-1} and $A_{i-1,\eta}, B_{i-1,\eta}$ for $\eta \in T'_i$. Consider the function $F: T'_i \to \aleph_{k_i}$ defined by $F(\eta) = \mathrm{Sup}(N_{\xi+1,\eta} \cap \aleph_{k_i})$. Then F has a value $< \aleph_{k_i}$. Since $k_i \notin A$, we have $m_i \neq k_i$, so F_{m_i} is $\aleph_{m_i}$-complete on a set of cardinality $\aleph_{m_i}$ so the assumptions of Lemma 4.1 hold. Hence there is $T''_{i+1} \subseteq T'_i$ such that $T''_{i+1} \in \mathcal{T}$ and $\mathrm{Sup}(F''T'_{i+1}) < \aleph_{k_i}$ by Lemma 4.1. Let $A_{i,\eta} = \mathrm{Suc}_{T''_{i+1}}(\eta)$ for $\eta \in T''_{i+1}$. Let $B_{i,\eta}$ be the move of the second player following his winning strategy in the game $GM_{\aleph_{n^*}}(F_{lh(\eta)})$. Let $T'_{i+1} \in \mathcal{T}$ be such that $B_{i,\eta} = \mathrm{Suc}_T(\eta)$ for each $\eta \in T$. Let α_i be such that $\mathrm{Sup}(F''T'_{i+1}) < \alpha_i < \aleph_{k_i}$.

Now, let $T'_{\xi+1} = \bigcap_{i<\omega} T'_i$ and $\alpha_{\xi+1,k_i} = \alpha_i$. Since $\langle A_{i,\eta}, B_{i,\eta}: i \in \omega\rangle$ is an initial segment of the play of the game $GM_{\aleph_{n^*}}(F_{lh(\eta)})$ with player two following his winning strategy, we have that $T'_{\xi+1} \in \mathcal{T}$. For each $\eta \in T'_{\xi+1}$, let $A_{\xi,\eta} = \mathrm{Suc}_{T'_{\xi+1}}(\eta)$. Note this is a legal move for the first player. Now player two chooses $B_{\xi,\eta}$ according to his winning strategy. Let $T_{\xi+1} \in \mathcal{T}$ be such that $B_{i,\eta} = \mathrm{Suc}_{T_{\xi+1}}(\eta)$ for each $\eta \in T_{\xi+1}$. This completes the construction as required. (Alternatively demand that in $\langle k_i: i < \omega\rangle$, each $m \in \{m < \omega: m \notin A, n^* < m < \omega\}$ appear ω many times and if $\xi = i \bmod \omega$, take care only of $\aleph_{k_i}$.)

Finally, let $T_{\aleph_{n^*}} = \bigcap_{\xi < \aleph_{n^*}} T_\xi$. Since $\langle A_{\xi,\eta}, B_{\xi,\eta}: \xi < \aleph_{n^*}\rangle$ is a play of the game $GM_{\aleph_{n^*}}(F_{lh(\eta)})$ with the second player following his winning strategy, it's easy to see that for each $\eta \in T_{\aleph_{n^*}}$ we have $|\mathrm{Suc}_{T_{\aleph_{n^*}}}(\eta)| = \aleph_{lh(\eta)}$. Now let b be an infinite branch of $T_{\aleph_{n^*}}$ such that $b(i) > \mathrm{Sup}(N_{\aleph_{n^*}, b\restriction i} \cap \aleph_{m_i})$, where $N_{\aleph_{n^*}, b\restriction i}$ is defined in the same way as $N_{\xi,\eta}$ was defined above. Such a branch b clearly exists.

Now, let $B = \mathrm{sk}^A(\{b(i): i \in \omega\} \cap \{\alpha_{\xi,m}: \xi < \aleph_{n^*} \wedge n^* < m \notin A\})$. Then for each $m \in A$, the set $\langle b(i): i < \omega \wedge m_i = m\rangle$ is cofinal in $B \cap \aleph_m$. Furthermore, for $n^* < m \notin A$, $\langle \alpha_{\xi,m}: \xi < \aleph_{n*}\rangle$ is cofinal in $B \cap \aleph_m$. Hence B is as required. ∎

THEOREM 4.2: *Assume* GCH. *Let* $0 \leq m < n^* < \omega$ *and* $E \subseteq \omega \backslash (n^*+1)$ *be such that for all* $i \in E, i+1 \notin E$ *and let* $j(i) = \mathrm{Max}(E \cap i)$. *Let* $\langle n_i: i \in E\rangle$ *with each* $n_i < n^*$. *Suppose that for each* $i \in E$, *there is an* $\aleph_i$*-complete filter* F_i *on* $\aleph_i$ *containing all clubs of* $\aleph_i$ *such that* $W_i = \{\alpha < \aleph_i: \mathrm{cf}(\alpha) = \aleph_{n_i}\} \in F_i^+$ *and the second player has a winning strategy in the game* $GM'_{\aleph_m,\aleph_i}(F_i)$.

Let $f: \omega \to n^*$ *be the function defined by*

$$f(i) = \begin{cases} n_i & \text{if } i \in E, \\ m & \text{if } n^* < i \notin E. \end{cases}$$

Then the set $S' = \{x \subset \aleph_\omega: |x| \leq \aleph_{n^*} \text{ and } (\forall i > n^*)\,\mathrm{cf}(x \cap \aleph_i) = \aleph_{f(i)}\}$ *is stationary in* $[\aleph_\omega]^{<\aleph_{n^*+1}}$.

Remarks: 1. Instead of GCH, it's enough to assume for $i < j$ in E that we have $(\aleph_{j-1})^{\aleph_i} = \aleph_{j-1}$.

2. The assumption is consistent, but not so if we strengthen it using $GM_{\aleph_m}(F_i)$. (By [Sh 542]).

Proof: Let $\lambda >> \aleph_\omega$ be a regular cardinal and $A = \langle \mathcal{H}(\lambda), \in, \lhd, < n_i: i \in E >, (\tau_i)_{i<\omega}, \ldots\rangle$ be a fully skolemized structure with skolem functions closed under compositions, where $\lhd$ is a well-ordering on $\mathcal{H}(\lambda)$. In order to prove the theorem, it suffices to show that there exists $B \prec A$ such that $|B| = \aleph_{n^*}$, $\mathrm{cf}(B \cap \aleph_i) = \aleph_{n_i}$ for each $i \in E$ and $\mathrm{cf}(B \cap \aleph_i) = \aleph_m$ for $n^* < i \notin E$.

For each $i > n^*$, let $h_i: W_i \to [\aleph_i]^{<\aleph_{n^*}}$ be defined by $h_i(\delta) = X_{i,\delta}$, where $X_{i,\delta}$ is the $\lhd$-least cofinal subset of δ of cardinality $\aleph_{n_i}$. Note that each h_i is definable in A.

We now define $\langle A_{i,0}, P_{i,\xi}, B_{i,\xi}: i \in E, \xi < \aleph_m\rangle$ and $\langle A_\xi: \xi \in \aleph_m\rangle$ by induction on $\xi < \aleph_m$ such that:

(1) for each $i \in E, \langle A_{i,0}, P_{i,\xi}, B_{i,\xi}: \xi < \aleph_m\rangle$ is a play of the game $GM'_{\aleph_m}(F_i)$ with the second player following his winning strategy;

(2) $A_{i,0} = W_i, A_0 = \mathrm{sk}^A(\{\emptyset\})$ and $A_\xi = \bigcup_{\xi'<\xi} A_{\xi'}$ if ξ is limit;

(3) $A_\xi \prec A, |A_\xi| < \aleph_{n^*}$ and $A_\xi \subseteq A_{\xi+1}$;

(4) for each $n^* < j < \omega, j \notin E$ we have $\sup(A_\xi \cap \aleph_j) < \sup(A_{\xi+1} \cap \aleph_j)$;

(5) for each $i \in E$, for all $\delta \in B_{i,\xi+1}$, we have $\mathrm{sk}^A(A_\xi \cup X_{i,\delta}) \cap \aleph_{i-1} \subseteq A_{\xi+1}$ and $\mathrm{sk}^A(A_\xi \cup \delta) \cap \aleph_i = \delta$.

We simulate the games $GM'_{\aleph_m,\aleph_i}(F_i)$ for $i \in E$ simultaneously. The first player chooses $A_{i,0} = W_i$ and then divides it into less than $\aleph_i$ parts for his (or her) 0^{th} move in the game $GM'_{\aleph_m,\aleph_i}(F_i)$. The second player always follows the winning strategy. For successor stage, suppose we have constructed $\langle B_{i,\xi}: i \in E\rangle$ and A_ξ. For $i \in E$, let $C_i = \{\alpha < \aleph_i: \mathrm{sk}^A(\alpha \cup A_\xi) \cap \aleph_i = \alpha\}$. Then C_i is a club in $\aleph_i$. So $C_i \in F_i$. For each $i \in E$, consider the function $f_i: B_{i,\xi} \cap C_i \to [\aleph_{i-1}]^{<\aleph_{n^*}}$ defined by $f_i(\delta) = \mathrm{sk}^A(A_\xi \cup X_{i,\delta}) \cap \aleph_{i-1}$. The first player divides $B_{i,\xi}$ into $\aleph_{i-1}$ parts as follows: $P_{i,\xi} = \{f_i^{-1}\{x\}: x \in [\aleph_{i-1}]^{<\aleph_{n^*}}\} \cup \{B_{i,\xi} \backslash C_i\}$. (Note that $|[\aleph_{i-1}]^{<\aleph_{n^*}}| = \aleph_{i-1}$ by GCH.) The second player chooses one of the parts, say $B_{i,\xi+1}$, according to his winning strategy. Note that the second player will not choose $B_{i,\xi}\backslash C_i$ as his move since $B_{i,\xi}\backslash C_i = \emptyset \bmod(F_i)$. (Otherwise he will lose right away.) So there must be some $X_i \in [\aleph_{i-1}]^{<\aleph_{n^*}}$ such that $f_i'' B_{i,\xi+1} = \{X_i\}$. Now let $X = \bigcup_{i\in E} X_i$ and $\alpha_j = \sup(A_\xi \cap \aleph_j)$ for $j \notin E$ and $A_{\xi+1} = \mathrm{sk}^A(A_\xi \cup X \cup \{\alpha_j: j \notin E\})$.

For limit stage, having defined $\langle B_{i,\xi'}: \xi' < \xi\rangle$, the first player just divides $\bigcap_{\xi'<\xi} B_{i,\xi'}$ into $\aleph_{i-1}$ parts anyway he wants. We let $B_{i,\xi}$ be the move of the second player following his winning strategy. This completes the construction and the sequences $\langle A_{i,0}, P_{i,\xi}, B_{i,\xi}: i \in E, \xi \in \aleph_m\rangle$ and $\langle A_\xi: \xi \in \aleph_m\rangle$ which clearly satisfy clauses (1)–(5) above.

Now, let $A^* = \bigcup_{\xi<\aleph_m} A_\xi$ and $W_i' = \bigcap_{\xi<\aleph_m} B_{i,\xi}$. Then $\mathrm{cf}(A^* \cap \aleph_j) = \aleph_m$ for $n^* < j \notin E$ by clause (4) above, and each W_i' is unbound in $\aleph_i$.

Let's enumerate E as $\langle i_n: n \in \omega\rangle$ in increasing order. We choose $\delta_{i_n} \in W'_{i_n}$. Let $B_n = \mathrm{sk}^A(A^* \cup \bigcup_{k\le n} X_{i_k,\delta_{i_k}})$ and $B_n' = \mathrm{sk}^A(A^* \cup X_{i_n,\delta_{i_n}})$. Then we have that $\sup(B_n' \cap \aleph_{i_n}) = \delta_{i_n}$ by clause (5) above. Also, we have that $B_n' \cap \aleph_{i_n-1} \subseteq A^*$ since if $\alpha \in B_n' \cap \aleph_{i_n-1}$ then $\alpha \in \mathrm{sk}^A(A_\xi \cup X_{i_n,\delta_{i_n}}) \cap \aleph_{i_n-1} \subseteq A_{\xi+1} \subseteq A^*$ for some $\xi < \aleph_m$.

Claim: *For all $n < \omega$, we have*

(a) $B_n \cap \aleph_{i_n-1} = B_{n-1} \cap \aleph_{i_n-1}$ *for* $n > 0$,

(b) $\sup(B_n \cap \aleph_{i_n}) = \delta_{i_n}$,

(c) $(\forall i_0 < j \notin E)[\sup(B_n \cap \aleph_j) = \sup(A^* \cap \aleph_j)]$.

Proof: To prove (a), it suffices to show that for any $\alpha \in B_n \cap \aleph_{i_n-1}, \alpha \in B_{n-1}$. Let $\alpha \in B_n \cap \aleph_{i_n-1}$. For simplicity, we may assume $\alpha = \tau(a^*, x_0, \ldots, x_{n-1}, x_n)$ for $a^* \in A^*, x_k \in X_{i_k,\delta_{i_k}}$ for $k \le n$ and for some skolem function τ.

Let $f\colon \prod_{k<n} \aleph_{i_k} \to \aleph_{i_n-1}$ be the function defined by letting $f(\vec{\beta}) = \tau(a^*, \vec{\beta}, x_n)$ if $\tau(a^*, \vec{\beta}, x_n) < \aleph_{i_n-1}$ and $f(\vec{\beta}) = 0$ otherwise. Then f is definable from a^* and x_n. So $f \in B'_n$.

Now, let $\vec{f} = \langle f_\xi \colon \xi < \aleph_{i_n-1}\rangle$ be a list of all the functions from $\prod_{k<n} \aleph_{i_k}$ to $\aleph_{i_n-1}$. (Note this is possible by GCH and $i_{n-1} < i_n - 1$.) By definability, we can choose $\vec{f} \in A^*$. But then $B'_n \models (\exists \xi < \aleph_{i_n-1}) f_\xi = f$. Let $\xi \in B'_n \cap \aleph_{i_n-1}$ be such that $f_\xi = f$. Then $\xi \in A^* \subseteq B_{n-1}$ since $B'_n \cap \aleph_{i_n-1} \subseteq A^*$. So $f = f_\xi \in B_{n-1}$. Therefore, $\alpha = f(x_0, \ldots, x_{n-1}) \in B_{n-1}$ since $x_k \in B_{n-1}$ for all $k < n$. We have thus proved part (a) of the claim.

Clause (b) follows from Lemma 2.1 and $\sup(B'_n \cap \aleph_{i_n}) = \delta_{i_n}$. By Lemma 2.1, $\sup(B_n \cap \aleph_{i_n}) = \sup(B'_n \cap \aleph_{i_n}) = \delta_{i_n}$.)

We prove (c) by induction on n. If $n = 0$, clause (c) follows from Lemma 2.1 and the choice of $A_{\xi+1}$ (and the α_j's) above. Now suppose (c) holds for $n-1$. We want to show (c) holds for n. By (a) and induction hypothesis, $\sup(B_n \cap \aleph_j) = \sup(B_{n-1} \cap \aleph_j) = \sup(A^* \cap \aleph_j)$ if $i_0 < j < i_n$ and $j \notin E$. For $i_n < j \notin E, \sup(B_n \cap \aleph_j) = \sup(B'_n \cap \aleph_j) = \sup(A^* \cap \aleph_j)$ by Lemma 2.1. This finishes the proof of the claim. ∎

We now can complete the proof of Theorem 4.2. Let $B^* = \bigcup_{n<\omega} B_n$. Then $B^* \prec A$ and $(\forall i \geq \min(E))[\mathrm{cf}(B^* \cap \aleph_i) = \aleph_{f(i)}]$ by the above claim. (Note that if $i > n^*$ and $i \leq i_n$, then $B^* \cap \aleph_i = B_n \cap \aleph_i$ by the claim.)

Finally, let $B = \mathrm{sk}^A(B^* \cup \aleph_{n^*})$. B is as required again by Lemma 2.1. So we have finished the proof of Theorem 4.2. ∎

5. Concluding remarks

The most natural context (at least for the second author) is having a constant cardinal κ, set $\mathfrak{a}$ of regular cardinals. Let $\lambda = \sup(\mathfrak{a})$ and we look for stationary subsets of $[\lambda]^{<\kappa}$. Let $\mathcal{F}^\kappa_{\mathfrak{a}} = \{f\colon f$ is a function with domain $\mathfrak{a}$ and $f(\theta)$ is a regular cardinal $< \theta$ and $\sup \mathrm{Rang}(f) < \kappa\}$. For an ideal J on $\mathfrak{a}$ and $f \in \mathcal{F}^\kappa_{\mathfrak{a}}$ we define $S^J_f = \{A \subseteq \lambda\colon |A| < \kappa$ and for some $\mathfrak{b} \in J$ for each $\theta \in \mathfrak{a}\backslash\mathfrak{b}$ the set $A \cap \theta$ is a bounded subset of θ with order type of cofinality $f(\theta)\}$. Note that $\mathfrak{a}, \lambda$ and κ can be reconstructed from f so we can just say "$S_f = S^J_f$ is stationary". We call the framework simple if $|\mathfrak{a}| < \kappa \leq \mathrm{Min}(\mathfrak{a})$, and we concentrate on it. If $J = \{\emptyset\}$ we may omit it.

$(*)_1$ If $\mathfrak{a}$ has a maximal element θ and $f \in \mathcal{F}^\kappa_{\mathfrak{a}}$ then $(f \restriction (\mathfrak{a}\backslash\{\theta\}) \in \mathcal{F}_{\mathfrak{a}\backslash\{\theta\}}$ and) S^J_f is stationary iff $S_{f\restriction(\mathfrak{a}\backslash\{\theta\})}$ is stationary.

$(*)_2$ For $f \in \mathcal{F}^\kappa_\mathfrak{a}$ and θ we have: S^J_f is stationary iff $S^J_{f\restriction(\mathfrak{a}\cap\theta)}$ is stationary and $S^J_{f\restriction(\mathfrak{a}\setminus\theta)}$ is stationary.

$(*)_3$ Assume $\mathfrak{a}$ has no last element and $f \in \mathcal{F}^\kappa_\mathfrak{a}$; then S_f is stationary iff (a) + (b) where:

(a) for every algebra M with universe $\sup(\mathfrak{a})$ for some $N \prec M$ we have: for every $\theta \in \mathfrak{a}$ large enough,

$$\mathrm{cf}(\sup(N \cap \theta) = f(\theta);$$

(b) for every $\theta \in \mathfrak{a}$ the set $S^J_{f\restriction(\mathfrak{a}\cap\theta)}$ is stationary.

$(*)_4$ Assume $n^* < \omega$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_{n^*}$ are members of $\mathrm{pcf}(\mathfrak{a})$ which are $> \sup(\mathfrak{a})$. Assume further $f \in \mathcal{F}^\kappa_\mathfrak{a}$, $\langle \mathfrak{b}_1, \ldots, \mathfrak{b}_{n^*}\rangle$ is a partition of $\mathfrak{a}$, and $f \restriction \mathfrak{b}_e$ is constant and $\lambda_e = \mathrm{tcf}(\prod \mathfrak{b}_e, <_{J\restriction\mathfrak{b}_e})$. Then S^J_f is stationary.

[Why? By the proof of Theorem 3.1. I.e. by $(*)_2$ used several times wlog $\mathrm{Min}(\mathfrak{a}) > |\mathfrak{a}|^{n^*+2}$; then by the proof of 3.1, if $1 \le i \le n^* \Rightarrow f(i) > |\mathfrak{a}|$ we succeed. Lastly use $(*)_3$ possibly several times.]

$(*)_5$ Assume

(a) $\mathfrak{b} \subseteq \mathfrak{a}$ is countable, $f \in \mathcal{F}^\kappa_\mathfrak{a}$, $f \restriction \mathfrak{b}$ is constantly $\aleph_0$ and $f \restriction (\mathfrak{a}\setminus\mathfrak{b})$ is constantly σ, δ is a unit ordinal of cofinably σ but $< \kappa$ and δ is divisible by $|\mathfrak{a}|$ (e.g. $\delta = \sigma$).

(b) For $\theta \in \mathfrak{a}$, the second player has a winning strategy in the game $GM_\delta(F_\theta, \theta \cap (\mathfrak{a}\setminus\mathfrak{b}))$ (see Definition 4.1(3)).

Then S_f is stationary.

[Why? Repeat the proof of Theorem 4.1, but we let $\langle\theta_i: i<\omega\rangle$ list $\mathfrak{b}$, each appearing infinitely often, and $\mathcal{T} = \{T: T$ subtree of $\bigcup_{e<\omega}\prod_{i<e}\theta_i$ such that for every $\eta \in T$ of length i we have $\{\alpha < \theta_i: \eta\hat{\ }\langle i\rangle \in T\} \neq \emptyset \bmod F_{\theta_i}\}$; let $\langle(\theta_\xi, m_\xi): \xi < \delta\rangle$ be such that: $\theta_\alpha \in \mathfrak{a}\setminus\mathfrak{b}, m_\alpha < \omega$, and each such pair occurs boundedly often. Then define the $T_\xi \in \mathcal{T}$ as in the proof of Theorem 4.1; in $T_{\xi+1}$ we take care of every $\eta \in T_{\xi+1}$ of length $\le m_\xi$.]

$(*)_6$ Assume

(a) $\mathfrak{b} \subseteq \mathfrak{a}$, $\sigma = \mathrm{cf}(\sigma) < \kappa, f \in \mathcal{F}^\kappa_\mathfrak{a}$, $f \restriction (\mathfrak{a}\setminus\mathfrak{b})$ is constantly σ, δ is an ordinal $< \kappa$ of cofinality σ and let $\sigma_\theta = [(\sup(\mathfrak{a}\cap\theta))^{\sigma+\sup(\mathfrak{b}\cap\theta)}]^+$.

(b) For $\theta \in \mathfrak{b}$, F_θ is a σ_θ-complete filter on θ extending the club filter such that player two has a winning strategy in the game $GM'_{\delta,\sigma_\theta}(F_\theta)$ (alternatively $GM'_\delta(F_\theta, \mathrm{pcf}(\mathfrak{a}\cap\theta))$).

Then S_f is stationary.

[Why? By the proof of Theorem 4.2.]

Improving $(*)_4$ we have:

$(*)_7$ Assume

(a) $\mathfrak{a} = \cup_{l \le n^*} \mathfrak{b}_l$,

(b) $\langle \mathrm{pcf}(\mathfrak{b}_l): l \le n^* \rangle$ is a sequence of pairwise disjoint sets,

(c) $f \in \mathcal{F}^\kappa_\mathfrak{a}$ and each $f \restriction \mathfrak{b}_l$ is constant.

[Why? We prove this by induction on $\max \mathrm{pcf}(\mathfrak{a}) = \max\{\max \mathrm{pcf}(\mathfrak{b}_l): l \le n^*\}$. The induction step is as in the proof of [Sh:g, VIII§1].]

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